

THE HEEGAARD DISTANCES COVER ALL NON-NEGATIVE INTEGERS

RUIFENG QIU, YANQING ZOU, QILONG GUO

ABSTRACT. (1) For any integers $n \geq 1$ and $g \geq 2$, there is a closed 3-manifold M_g^n which admits a distance n Heegaard splitting of genus g except that the pair of (g, n) is $(2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic except that the pair of (g, n) is $(3, 1)$. (2) For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many non-homeomorphic closed 3-manifolds admitting distance n Heegaard splittings of genus g .

Keywords: Attaching Handlebody, Heegaard distance, Subsurface Projection.

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1. INTRODUCTION

Let S be a compact surface with $\chi(S) \leq -2$ not a 4-punctured sphere. Harvey [8] defined the curve complex $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S , and $k+1$ distinct vertices x_0, x_1, \dots, x_k determine a k -simplex of $\mathcal{C}(S)$ if and only if they are represented by pairwise disjoint simple closed curves. For two vertices x and y of $\mathcal{C}(S)$, the distance of x and y , denoted by $d_{\mathcal{C}(S)}(x, y)$, is defined to be the minimal number of 1-simplexes in a simplicial path joining x to y . In other words, $d_{\mathcal{C}(S)}(x, y)$ is the smallest integer $n \geq 0$ such that there is a sequence of vertices $x_0 = x, \dots, x_n = y$ such that x_{i-1} and x_i are represented by two disjoint essential simple closed curves on S for each $1 \leq i \leq n$. For two sets of vertices in $\mathcal{C}(S)$, say X and Y , $d_{\mathcal{C}(S)}(X, Y)$ is defined to be $\min\{d_{\mathcal{C}(S)}(x, y) \mid x \in X, y \in Y\}$. Now let S be a torus or a once-punctured torus. In this case, Masur and Minsky [28] define $\mathcal{C}(S)$ as follows: The vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on S , and $k+1$ distinct vertices x_0, x_1, \dots, x_k determine a k -simplex of $\mathcal{C}(S)$ if and only if x_i and x_j are represented by two simple closed curves c_i and c_j on S such that c_i intersects c_j in just one point for each $0 \leq i \neq j \leq k$.

Let M be a compact orientable 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$, then we say M has a Heegaard splitting, denoted by $M = V \cup_S W$, where $\partial_+ V$ (resp. $\partial_+ W$) means the positive boundary of V (resp. W). We denote by $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) the set of vertices in $\mathcal{C}(S)$ such that each element of $\mathcal{D}(V)$ (resp. $\mathcal{D}(W)$) is represented by the boundary of an essential disk in V (resp. W). The distance of the Heegaard splitting $V \cup_S W$, denoted by $d(S)$, is defined to be $d_{\mathcal{C}(S)}(\mathcal{D}(V), \mathcal{D}(W))$. See [9].

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It is well known that a 3-manifold admitting a high distance Heegaard splitting has good topological and geometric properties. For example, Hartshorn[10] and Scharlemann[33] showed that a 3-manifold admitting a high distance Heegaard splitting contains no essential surface with small Euler characteristic number; Scharlemann and Tomova[34] showed that a high distance Heegaard splitting is a unique minimal Heegaard splitting up to isotopy. By Geometrization theorem and Hempel's work in [9], a 3-manifold M admitting a distance at least 3 Heegaard splitting is hyperbolic. From this view, studying Heegaard distance is an active topic on Heegaard splitting. The following is a brief survey on the existences of high distance Heegaard splittings:

Hempel[9] showed that for any integers $g \geq 2$, and $n \geq 2$, there is a 3-manifold which admitting a distance at least n Heegaard splitting of genus g . Similar results are obtained in different ways by [4] and [6]. Minsky, Moriah and Schleimer [30] proved the same result for knot complements, and Li[22] constructed the non-Haken manifolds admitting high distance Heegaard splittings. In general, generic Heegaard splittings have Heegaard distances at least n for any $n \geq 2$, see [23],[24],[25]. By studying Dehn filling, Ma, Qiu and Zou[32] proved that distances of genus 2 Heegaard splittings cover all non-negative integers except 1. Recently, Ido, Jang and Kobayashi[11] proved that, for any $n > 1$ and $g > 1$, there is a compact 3-manifold with two boundary components which admits a distance n Heegaard splitting of genus g . And Johnson[14] proved that there are always existing closed 3-manifold admitting a distance $n \geq 5$, genus g Heegaard splitting.

The main result of this paper is the following:

Theorem 1. For any integers $n \geq 1$ and $g \geq 2$, there is a closed 3-manifold M_g^n which admits a distance n Heegaard splitting of genus g except that the pair of (g, n) is $(2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic except that the pair of (g, n) is $(3, 1)$.

Remark on Theorem 1. (1) It is well known that there is not a distance 1 Heegaard splitting of genus 2.

(2) By the above argument, a 3-manifold admitting a distance at least 3 Heegaard splitting is hyperbolic. Hempel [9] showed that any Heegaard splitting of a Seifert 3-manifold has distance at most 2. Now a natural question is: For any integer $g \geq 2$, is there a closed hyperbolic 3-manifold admitting a distance 2 Heegaard splitting of genus g ? Suppose first that $g = 2$. Eudave-Munoz[5] proved that there is a hyperbolic $(1, 1)$ -knot in 3-sphere, say K . In this case, the complement of K , say M_K , admits a distance 2 Heegaard splitting of genus 2. By the main results in [1], [14] and [31], there is a slope r on ∂M_K such that the manifold obtained by doing a surgery on M_K along r , say $M_K(r)$, is still hyperbolic. Hence $M_K(r)$ admits a distance 2 Heegaard splitting of genus 2. Maybe the answer to this question has been well known when $g \geq 3$. However we did not find published papers related to it.

(3) If M admits a distance 1 Heegaard splitting of genus 3, then M contains an essential torus. Hence M is not hyperbolic.

(4) The proof of Theorem 1 implies the following fact:

Let n be a positive integer, $\{F_1, \dots, F_n\}$ be a collection of closed orientable surfaces, and I and $J = \{1, \dots, n\} \setminus I$ be two subsets of $\{1, \dots, n\}$. Then, for any integers

$g \geq \max\{\sum_{i \in I} g(F_i), \sum_{j \in J} g(F_j)\}$ and $m \geq 2$, there is a compact 3-manifold M admitting a distance m Heegaard splitting of genus g , say $M = V \cup_S W$, such that $F_i \subset \partial_- V$ for $i \in I$, and $F_j \subset \partial_- W$ for $j \in J$. We omit the proof.

Under the arguments in Theorem 1, we have the following result:

Theorem 2. For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many non-homeomorphic closed 3-manifolds admitting distance n Heegaard splittings of genus g .

We organize this paper as follows:

Section 2 is devoted to introduce some results on curve complex. Then we will prove Theorem 1 when $n \neq 2$ in Section 3, Theorem 1 when $n = 2$ in Section 5, and Theorem 2 in Section 4.

2. PRELIMINARIES OF CURVE COMPLEX

Let S be a compact surface of genus at least 1, and $\mathcal{C}(S)$ be the curve complex of S . We call a simple closed curve c in S is essential if c bounds no disk in S and is not parallel to ∂S . Hence each vertex of $\mathcal{C}(S)$ is represented by the isotopy class of an essential simple closed curve in S . For simplicity, we do not distinguish the essential simple closed curve c and its isotopy class c without any further notation. The following lemma is well known, see [27], [28], [29].

Lemma 2.1. $\mathcal{C}(S)$ is connected, and the diameter of $\mathcal{C}(S)$ is infinite.

We call a collection $\mathcal{G} = \{a_0, a_1, \dots, a_n\}$ is a geodesic in $\mathcal{C}(S)$ if each $a_i \subset \mathcal{C}^0(S)$ and $d_{\mathcal{C}(S)}(a_i, a_j) = |i - j|$, for any $0 \leq i, j \leq n$. And the length of \mathcal{G} is denoted by $\mathcal{L}(\mathcal{G})$ is defined to be n . By the connection of $\mathcal{C}^1(S)$, there is always a shortest path in $\mathcal{C}^1(S)$ connecting any two vertices of $\mathcal{C}(S)$. Thus for any two distance n vertices α, β , we call a geodesic \mathcal{G} connecting α, β if $\mathcal{G} = \{a_0 = \alpha, \dots, a_n = \beta\}$. Now for any two sub-simplicial complex $X, Y \subset \mathcal{C}(S)$, we call a geodesic \mathcal{G} realizing the distance of X and Y if \mathcal{G} connecting an element $\alpha \in X$ and an element $\beta \in Y$ such that $\mathcal{L}(\mathcal{G}) = d_{\mathcal{C}(S)}(X, Y)$.

Let F be a compact surface of genus at least 1 with non-empty boundary. Similar to the definition of the curve complex $\mathcal{C}(F)$, we can define the arc and curve complex $\mathcal{AC}(F)$ as follows:

Each vertex of $\mathcal{AC}(F)$ is the isotopy class of an essential simple closed curve or an essential properly embedded arc in F , and a set of vertices form a simplex of $\mathcal{AC}(F)$ if these vertices are represented by pairwise disjoint arcs or curves in F . For any two disjoint vertices, we place an edge between them. All the vertices and edges form 1-skeleton of $\mathcal{AC}(F)$, denoted by $\mathcal{AC}^1(F)$. And for each edge, we assign it length 1. Thus for any two vertices α and β in $\mathcal{AC}^1(F)$, the distance $d_{\mathcal{AC}(F)}(\alpha, \beta)$ is defined to be the minimal length of paths in $\mathcal{AC}^1(F)$ connecting α and β . Similarly, we can define the geodesic in $\mathcal{AC}(F)$.

When F is a subsurface of S , we call F is essential in S if the induced map of the inclusion from $\pi_1(F)$ to $\pi_1(S)$ is injective. Furthermore, we call F is a proper essential subsurface of S if F is essential in S and at least one boundary component of F is essential in S . For more details, see [29].

So if F is an essential subsurface of S , there is some connection between the $\mathcal{AC}(F)$ and $\mathcal{C}(S)$. For any $\alpha \in \mathcal{C}^0(S)$, there is a representative essential simple closed curve α_{geo} such that the intersection number $i(\alpha_{geo}, \partial F)$ is minimal. Hence each component of $\alpha_{geo} \cap F$ is essential in F or $S - F$. Now for $\alpha \in \mathcal{C}(S)$, let $\kappa_F(\alpha)$ be isotopy classes of the essential components of $\alpha_{geo} \cap F$. It is well defined since for any two isotopy class α_1 and α_2 of α which both intersect ∂F minimally, either

(1) $\alpha_1 \cap \alpha_2 = \emptyset$. Then they bounds an annulus A in S . Hence either

(1.1) $\alpha_1 \cap \partial F = \emptyset$. If α_1 is essential in F , then $A \cap F = \emptyset$. Hence $A \subset F$. And $\alpha_1 \cap F$ is isotopic to $\alpha_2 \cap F$ in F . If both $\alpha_1 \cap F$ and $\alpha_2 \cap F$ are inessential, then the essential components of $\alpha_1 \cap F$ and $\alpha_2 \cap F$ are \emptyset . Or,

(1.2) $\alpha_1 \cap \partial F \neq \emptyset$.

By minimality of intersection number, $A \cap \partial F$ consists of squares. It is not hard to see that each component $\alpha_1 \cap F$ (resp. $\alpha_2 \cap F$) is essential. And each component $\alpha_1 \cap F$ is isotopic to one component of $\alpha_2 \cap F$. The reverse is true. Or,

(2) $\alpha_1 \cap \alpha_2 \neq \emptyset$. Since the intersection number $i(\alpha_1, \alpha_2) = 0$, by Bigon Criterion (proposition 1.7[7]), there is always an innermost Bigon B bounded by $\alpha_1 \cup \alpha_2$ in S . Since there is no proper bigon in B bounded by ∂B and ∂F (the minimality of $\alpha_1 \cap \partial F$ and $\alpha_2 \cap \partial F$), we can isotopy α_1 and α_2 such that $\alpha'_1 \cup \alpha$ (resp. $\alpha'_2 \cup \alpha_2$) bounds an annulus in S and $\alpha'_1 \cap \partial F$ (resp. $\partial F \cap \alpha'_2$) is minimal. And the Bigon B is vanishing. Following (1), we get that any essential component of $\alpha_1 \cap F$ (resp. $\alpha_2 \cap F$) is isotopic to an essential component of $\alpha'_1 \cap F$ (resp. $\alpha'_2 \cap F$). And the reverse is true. We can do it again and again until there is no such Bigon. Then it returns to (1). \triangleleft

For any $\gamma \in \mathcal{C}(F)$, $\gamma' \in \sigma_F(\beta)$ if and only if γ' is the essential boundary component of a closed regular neighborhood of $\gamma \cup \partial F$. Specially, let $\sigma_F(\emptyset) = \emptyset$. Now let $\pi_F = \sigma_F \circ \kappa_F$. Then the map π_F links the $\mathcal{AC}(F)$ and $\mathcal{C}(S)$, which is the defined subsurface projection map in [29].

We say $\alpha \in \mathcal{C}^0(S)$ cuts F if $\pi_F(\alpha) \neq \emptyset$. If $\alpha, \beta \in \mathcal{C}^0(S)$ both cut F , we write $d_{\mathcal{C}(F)}(\alpha, \beta) = \text{diam}_{\mathcal{C}(F)}(\pi_F(\alpha), \pi_F(\beta))$. And if $d_{\mathcal{C}(S)}(\alpha, \beta) = 1$, then $d_{\mathcal{AC}(F)}(\alpha, \beta) \leq 1$ and $d_{\mathcal{C}(F)}(\alpha, \beta) \leq 2$, observed by H.Masur and Y.N.Minsky at first. What if the two vertices α and β has distance k in $\mathcal{C}(S)$?

The following is immediately followed from the above observation.

Lemma 2.2. Let F and S be as above, $\mathcal{G} = \{\alpha_0, \dots, \alpha_k\}$ be a geodesic of $\mathcal{C}(S)$ such that α_j cuts F for each $0 \leq i \leq k$. Then $d_{\mathcal{C}(F)}(\alpha_0, \alpha_k) \leq 2k$.

However, when k is quite large, the Lemma 2.2 can not provide more information. In general, Masur-Minsky [29] proved the following result called Bounded Geodesic Image Theorem.

Lemma 2.3. Let F be an essential sub-surface of S , and γ be a geodesic segment in $\mathcal{C}(S)$, such that $\pi_F(v) \neq \emptyset$ for every vertex v of γ . Then there is a constant \mathcal{M} depending only on S so that $\text{diam}_{\mathcal{C}(F)}(\pi_F(\gamma)) \leq \mathcal{M}$.

When S is closed with $g(S) \geq 2$, there is always a compact 3-manifold M with S as its compressible boundary. Let $\mathcal{D}(M, S)$, called disk set for S , be the subset of vertices of $\mathcal{C}(S)$, where each element bounds a disk in M . Now an essential simple closed curve on S , say c , is said to be disk-busting if $S - c$ is incompressible in M . Since any two essential disks intersect in a typical way, it provides more

information to study the subsurface projection of disk complex. The following Disk Image Theorem is proved by T.Li [18], H.Masur and S.Schleimer [31] independently.

Lemma 2.4. Let M be a compact orientable and irreducible 3-manifold. S is a boundary component of M . Suppose $\partial M - S$ is incompressible. Let \mathcal{D} be the disk complex of S , $F \subset S$ be an essential subsurface. Assume each component of ∂F is disk-busting. Then either

(1) M is an I-bundle over some compact surface, F is a horizontal boundary of the I-bundle and the vertical boundary of this I-bundle is a single annulus. Or,

(2) The image of this complex, $\kappa_F(\mathcal{D})$, lies in a ball of radius 3 in $\mathcal{AC}(F)$. In particular, $\kappa_F(\mathcal{D})$ has diameter 6 in $\mathcal{AC}(F)$. Moreover, $\pi_F(\mathcal{D})$ has diameter at most 12 in $\mathcal{C}(F)$.

Note. For any I-bundle J over a bounded compact surface P , $\partial J = \partial_v J \cup \partial_h J$, where the vertical boundary $\partial_v J$ is the I-bundle related to ∂P , and the horizontal boundary $\partial_h J$ is the portion of ∂J transverse to the I-fibers.

On the other side, J.Hempel [9] defined a full simplex X on S to be a dimension $3g(S) - 4$ simplex in $\mathcal{C}(S)$. Hence, after attaching 2-handles and 3-handles along the vertices of X in the same side of S from the same side, we can get a handlebody, denoted by H_X .

Lemma 2.5 [9]. Let S be a closed, orientable surface of genus at least 2. For any positive number d , any full simplex X of $\mathcal{C}(S)$, there is another full simplex Y of $\mathcal{C}(S)$ such that $d_{\mathcal{C}(S)}(\mathcal{D}(H_X), \mathcal{D}(H_Y)) \geq d$.

Through subsurface projection, the Bounded Geodesic Image theorem links the geodesic in curve complex and a proper subsurface. The example 1.5 [29] shows that there is a geometry rigidity in curve complex. With the property of infinity of diameter of curve complex, we can construct any long geodesic in curve complex. Furthermore, we also require that the constructed geodesic satisfying some condition, such as the first and last vertices are represented by separating essential simple closed curves.

We organize our results as the following lemma which is a more stronger version of Lemma 4.1 in [32].

Lemma 2.6. Let g, n, m, s, t be five integers such that $g, m, n \geq 2$, and $1 \leq t, s \leq g - 1$. Let S_g be a closed surface of genus g . Then there are two essential separating curves α and β in S_g such that $d_{\mathcal{C}(S_g)}(\alpha, \beta) = n$, one component of $S_g - \alpha$ has genus t while one component of $S_g - \beta$ has genus s . Furthermore, there is a geodesic $\mathcal{G} = \{a_0 = \alpha, a_1, \dots, a_{n-1}, a_n = \beta\}$ in $\mathcal{C}(S_g)$ such that

- (1) a_i is non-separating in S_g for $1 \leq i \leq n - 1$, and
- (2) $m\mathcal{M} + 2 \leq d_{\mathcal{C}(S^{a_i})}(a_{i-1}, a_{i+1}) = m\mathcal{M} + 6$, where S^{a_i} is the surface $S - N(a_i)$ for $1 \leq i \leq n - 1$.

Proof. Let α be an essential separating curve in S such that one component of $S_g - \alpha$, say S_1 , has genus t .

Suppose first that $n = 2$. Let b be a non-separating curve in S_g which is disjoint from α . Let S^b be the surface $S_g - N(b)$, where $N(b)$ is a open regular neighborhood of b in S_g . Then S^b is a genus $g - 1$ surface with two boundary components. Furthermore, α is an essential separating simple closed curve in S^b .

By Lemma 2.1, $\mathcal{C}^1(S^b)$ is connected and its diameter is infinite. Hence there is an essential simple closed curve c in S^b with $d_{\mathcal{C}(S^b)}(\alpha, c) = m\mathcal{M} + 4$. Note that $g - 1 \geq 1$. If c is separating in S^b , there is a non-separating essential simple closed curve c^* in S^b such that $c \cap c^* = \emptyset$. Hence $d_{\mathcal{C}(S^b)}(c, c^*) = 1$, and $m\mathcal{M} + 3 \leq d_{\mathcal{C}_{S^b}}(\alpha, c^*) \leq m\mathcal{M} + 5$. It means there is a non-separating slope c in S^b such that $m\mathcal{M} + 3 \leq d_{\mathcal{C}_{S^b}}(\alpha, c) \leq m\mathcal{M} + 5$.

Let l be a non-separating simple closed curve in S^b such that l intersects c in one point, and e be the boundary of the regular neighborhood of $c \cup l$. Then e bounds a once-punctured torus T containing l and c . Since $s \leq g - 1$, there is an essential separating simple closed curve β such that β bounds a once-punctured surface of genus s containing T as a sub-surface. See Figure 1.

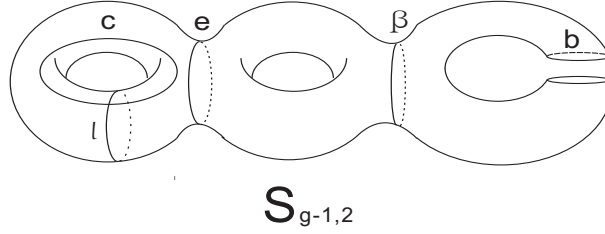


Figure 1

It is easy to see that β is also separating in S_g . Now we prove that $d_{\mathcal{C}(S_g)}(\alpha, \beta) = 2$, and $d_{\mathcal{C}(S_g)}(\alpha, c) = 2$.

Since $\alpha \cap b = \emptyset$, $\beta \cap b = \emptyset$ and $c \cap b = \emptyset$, $d_{\mathcal{C}(S_g)}(\alpha, \beta) \leq 2$, and $d_{\mathcal{C}(S_g)}(\alpha, c) \leq 2$. Since $c \cap \beta = \emptyset$, by the assumption on $d_{\mathcal{C}(S^b)}(\alpha, c)$, $m\mathcal{M} + 2 \leq d_{\mathcal{C}(S^b)}(\alpha, \beta) \leq m\mathcal{M} + 6$. Hence $d_{\mathcal{C}(S_g)}(\beta, \alpha) = 2$. For if $d_{\mathcal{C}(S_g)}(\alpha, \beta) \leq 1$, then, by Lemma 2.3, $d_{\mathcal{C}(S^b)}(\alpha, \beta) \leq \mathcal{M}$, a contradiction. Similarly, $d_{\mathcal{C}(S_g)}(\alpha, c) = 2$. And $\mathcal{G} = \{a_0 = \alpha, a_1 = b, a_2 = \beta\}$ and $\mathcal{G}^* = \{a_0 = \alpha, a_1 = b, a_2 = c\}$ are two geodesics of $\mathcal{C}(S_g)$. Furthermore, \mathcal{G} satisfies the conclusion of Lemma 2.6.

Now we prove Lemma 2.6 by induction on n .

Assumption. Let $k \geq 2$. Suppose that there are two essential separating simple closed curves α and β , and a non-separating simple closed curve c in S_g such that $d_{\mathcal{C}(S_g)}(\alpha, \beta) = k$, $d_{\mathcal{C}(S_g)}(\alpha, c) = k$, and one component of $S_g - \alpha$ has genus t while one component of $S_g - \beta$ has genus s . Furthermore, there is a geodesic $\mathcal{G} = \{\alpha = a_0, a_1, \dots, a_{k-1}, a_k = \beta\}$ satisfying Lemma 2.6(1) and (2), and a geodesic $\mathcal{G}^* = \{\alpha, a_1, \dots, a_{k-1}, c\}$ is also a geodesic connecting α to c satisfying $m\mathcal{M} + 3 \leq d_{\mathcal{C}(S^{a_i})}(a_{i-1}, a_{i+1}) \leq m\mathcal{M} + 5$, for any $1 \leq i \leq k - 2$ and $m\mathcal{M} + 3 \leq d_{\mathcal{C}(S^{a_{k-1}})}(a_{k-2}, c) \leq m\mathcal{M} + 5$,

Let S^c be the surface $S_g - N(c)$, where $N(c)$ is a open regular neighborhood of c in S_g . Since c is non-separating in S_g , S^c is a genus $g - 1$ surface with two boundary components. Since $\mathcal{G}^* = \{\alpha, a_1, \dots, a_{k-1}, c\}$ is also a geodesic connecting α to c , a_{k-1} is an essential non-separating simple closed curve in S^c . By the above argument, there are a non-separating curve h and a separating curve e in S^c such that e bounds a once-punctured torus T^* containing h , $m\mathcal{M} + 3 \leq d_{\mathcal{C}(S^c)}(h, a_{k-1}) \leq m\mathcal{M} + 5$, and $m\mathcal{M} + 2 \leq d_{\mathcal{C}(S^c)}(e, a_{k-1}) \leq m\mathcal{M} + 6$. And there is also an essential separating simple closed curve γ which bounds a genus s sub-surface of S^c containing T^* as a

sub-surface. Not hard to see γ is also separating in S_g . Since h is disjoint from γ , $m\mathcal{M} + 2 \leq d_{C(S^c)}(\gamma, a_{k-1}) \leq m\mathcal{M} + 6$.

Now we prove that $d_{C(S_g)}(\alpha, h) = k + 1$, $d_{C(S_g)}(\alpha, \gamma) = k + 1$.

Suppose, otherwise, that $d_{C(S_g)}(\alpha, h) = x \leq k$. Then there exists a geodesic line $\mathcal{G}_1 = \{\alpha = b_0, \dots, b_x = h\}$. Note that each of α and h is not isotopic to c . with the length is less than or equal to K . Since $d_{C(S_g)}(\alpha, c) = k$, b_j is not isotopic to c for $1 \leq j \leq x - 1$. That means b_j cuts S^c for each $0 \leq j \leq x$. By Lemma 2.3, $d_{S^c}(\alpha, h) \leq \mathcal{M}$. On the other side, since $d_{C(S_g)}(\alpha, c) = k$, a_j is not isotopic to c for $0 \leq j \leq k - 1$. By using Lemma 2.3 again, $d_{S^c}(\alpha, a_{k-1}) \leq \mathcal{M}$. Then $d_{C(S^c)}(a_{k-1}, h) \leq 2\mathcal{M}$. It contradicts the choice of h .

Now $\mathcal{G}' = \{a_0 = \alpha, a_1, \dots, a_{k-1}, c, \gamma\}$ and $\mathcal{G}'' = \{a_0 = \alpha, a_1, \dots, a_{k-1}, c, h\}$ are two geodesics satisfying the Assumption. Hence Lemma 2.6 holds. END.

3. PROOF OF THEOREM 1 (1)

In this section, we will prove the following proposition:

Proposition 3.1. For any positive integers $n \neq 2$ and $g \geq 2$, there is a closed 3-manifold which admits a distance n Heegaard splitting of genus g except that the pair of (g, n) is $(2, 1)$. Furthermore, M_g^n can be chosen to be hyperbolic except that the pair of (g, n) is $(3, 1)$.

Proof. We first suppose that $n \geq 3$.

Let S be a closed surface of genus g . By Lemma 2.6, there are two separating essential simple closed curves α and β such that $d_{C(S)}(\alpha, \beta) = n$ for $n \geq 3$. Let V be the compression body obtained by attaching a 2-handle to $S \times [0, 1]$ along $\alpha \times \{1\}$, and W be the compression body obtained by attaching a 2-handle to $S \times [-1, 0]$ along $\beta \times \{-1\}$. Then $V \cup_S W$ is a Heegaard splitting where S is the surface $S \times \{0\}$, see Figure 2. Since V contains only one essential disk B with $\partial B = \alpha$ up to isotopy, and W contains only one disk D with $\partial D = \beta$ up to isotopy, $d_{C(S)}(V, W) = n$.

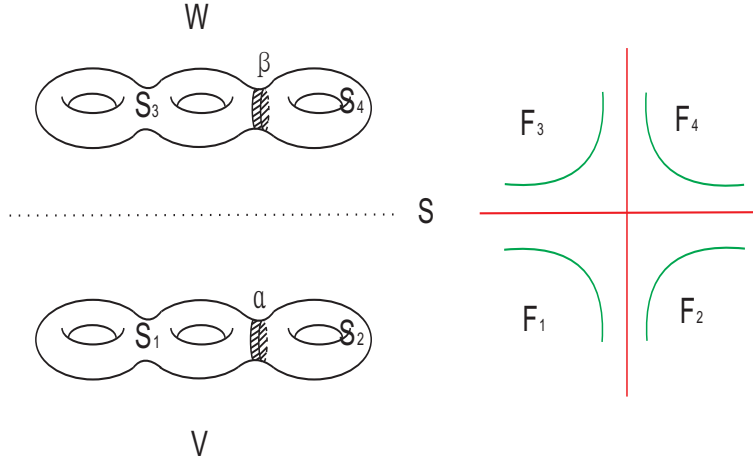


Figure 2

Let F_1 and F_2 be the components of $\partial_- V$, and S_1 and S_2 be the two components of $S - \alpha$. Similarly, let F_3 and F_4 be the components of $\partial_- W$, and S_3 and S_4 be the two components of $S - \beta$. Now B cuts V into two manifolds $F_1 \times I$ and $F_2 \times I$, and D cuts W into two manifolds $F_3 \times I$ and $F_4 \times I$. See Figure 2. By Lemma 2.6, we may assume that S_3 is a once-punctured torus.

We first consider the compression body V . We may assume that $F_i = F_i \times \{0\}$, $S_i \cup B = F_i \times \{1\}$ for $1 \leq i \leq 2$. Let $f_{F_i} : S_i \cup B \rightarrow F_i$ be the natural homeomorphism such that $f_{F_i}(x \times \{1\}) = x \times \{0\}$ for $i = 1, 2$ and $f_{F_i}(\emptyset) = \emptyset$. No doubt that f_{F_i} is well defined. Then, for any two essential simple closed curves $\zeta, \theta \subset S_i \cup B$, $d_{\mathcal{C}(F_i)}(f(\zeta), f(\theta)) = d_{\mathcal{C}(S_i \cup B)}(\zeta, \theta)$ for $i = 1, 2$. See Figure 3. Hence f_{F_i} induces an isomorphism from $\mathcal{C}(S_i \cup B)$ to $\mathcal{C}(F_i)$, for any $i = 1, 2$. Denote the isomorphism by f_{F_i} too. Note that the shadow disk in Figure 3 is B .

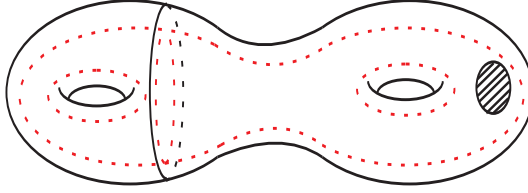


Figure 3

Let $\iota : S_i \rightarrow S_i \cup B$ be the inclusion map for $i = 1, 2$. Note that ∂S_i contains only one component. If c is an essential simple closed curve in S_i , $\iota(c)$ is also essential in $S_i \cup B$. Now, for any two essential simple closed curves $\zeta, \theta \subset S_i$, $d_{\mathcal{C}(S_i \cup B)}(\iota(\zeta), \iota(\theta)) \leq d_{S_i}(\zeta, \theta)$ for $i = 1, 2$. Hence ι induces a distance non-increasing map from $\mathcal{C}(S_i)$ to $\mathcal{C}(S_i \cup B)$, for any $i = 1, 2$. Denote the inclusion map by ι too. Then we can define a projection map :

$$\psi_{F_i} = f_{F_i} \circ \iota \circ \pi_{S_i} : \mathcal{C}^0(S) \rightarrow \mathcal{C}^0(F_i).$$

Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n \geq 2$, $\alpha \cap \beta \neq \emptyset$. By the argument in Section 2, $\text{diam}_{S_i}(\pi_{S_i}(\beta)) \leq 2$. Hence $\text{diam}_{\mathcal{F}_i}(\psi_{F_i}(\beta)) \leq 2$.

We start to attach a handlebody to V along F_1 . Then either

(1) F_1 is a torus. By Lemma 2.1, there is an essential simple closed curve r in F_1 such that $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\beta), r) \geq \mathcal{M} + 1$. let J_r be a solid torus such that $\partial J_r = F_1$, and r bounds a disk in J_r . In this case, J_r contains only one essential disk up to isotopy. Let V_{F_1} be the manifold $V \cup J_r$. Or,

(2) $g(F_1) \geq 2$. By Lemma 2.5, there is a full simplex X on of $\mathcal{C}(F_1)$ such that $d_{\mathcal{C}(F_1)}(\mathcal{D}(H_X), \psi_{F_1}(\beta)) \geq \mathcal{M} + 1$, where H_X is the handlebody obtained by attaching 2-handles to F_1 along X then 3-handles to cap off the possible 2-spheres. In this case, we denote by V_{F_1} the manifold $V \cup H_X$.

In whole words, V_{F_1} is a compression body with only one minus boundary component F_2 . See Figure 4. Hence $V_{F_1} \cup_S W$ is a Heegaard splitting.

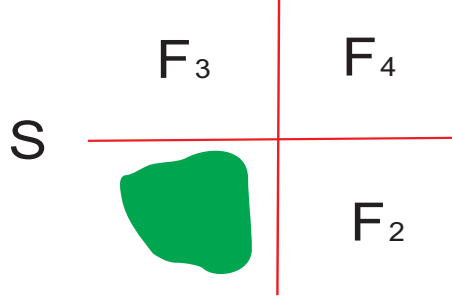


Figure 4

Claim 3.2. The Heegaard distance of $V_{F_1} \cup_S W$, say $d_{\mathcal{C}(S)}(V_{F_1}, W)$, is n .

Proof. Suppose, otherwise, that $d_{\mathcal{C}(S)}(V_{F_1}, W) = k < n$. Since W contains only one essential disk D up to isotopy such that $\partial D = \beta$, there is an essential disk B_1 in V_{F_1} such that $d_{\mathcal{C}(S)}(\partial B_1, \beta) = k \leq n - 1$, i.e, there is a geodesic $\mathcal{G} = \{a_0 = \beta, \dots, a_k = \partial B_1\}$, where $k \leq n - 1$.

Fact 3.3. $a_j \cap S_1 \neq \emptyset$, for any $0 \leq j \leq k$.

Suppose that $a_j \cap S_1 = \emptyset$ for some $0 \leq j \leq k$. $j \neq k$ since $a_k = \partial B_1$, and $\partial S_1 = \alpha$ and if $a_k \cap S_1 = \emptyset$, then $B_1 \subset F_2 \times I$, and B_1 is inessential in V_{F_1} . $j \neq 0$ since $a_0 = \beta$. Hence there is a geodesic $\mathcal{G}^* = \{\beta = a_0, \dots, a_j, \alpha\}$. It means that $d_{\mathcal{C}(S)}(\alpha, \beta) \leq k < n$, a contradiction.

By Lemma 2.3, $d_{\mathcal{C}(S_1 \cup B)}(\partial B_1, \beta) \leq \mathcal{M}$. Furthermore, $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\partial B_1), \psi_{F_1}(\beta)) \leq \mathcal{M}$. Depending on the way of intersection between B_1 and B , either

(1) $B_1 \cap B = \emptyset$. Since B_1 is not isotopic to B , $\psi_{F_1}(\partial B_1)$ bounds an essential disk in H_X or J_r depending on $g(F_1)$, where H_X and J_r are constructed as above. It contradicts the choice of X or r . Or,

(2) $B_1 \cap B \neq \emptyset$. Let a be an outermost arc of $B_1 \cap B$ on B_1 . It means that a , together with a sub-arc $\gamma \subset \partial B_1$, bounds a disk B_γ such that $B_\gamma \cap B = a$. Since B cuts V_{F_1} into a handlebody H which contains F_1 and a I -bundle $F_2 \times I$, $B_\gamma \subset H$. Hence $\psi_{F_1}(\partial B_1)$ bounds an essential disk in H_X or J_r . By the argument in (1), it is impossible. END.(Claim 1)

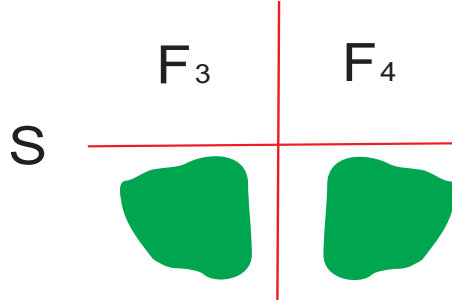


Figure 5

Now V_{F_1} is a compression body which has only one minus boundary component F_2 . Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n \geq 2$, $\beta \cap S_2 \neq \emptyset$. By Lemma 2.6, there is always a

full simplex Y on F_2 such that $d_{\mathcal{C}(F_2)}(\mathcal{D}(H_Y), \psi_{F_2}(\beta)) \geq \mathcal{M} + 1$, where H_Y is the handlebody obtained by attaching 2-handles to F_2 along Y then 3-handles to cap off the possible 2-spheres, and ψ_{F_2} is defined as before. Let V_{F_1, F_2} be the manifold obtained by attaching H_Y to V_{F_1} along F_2 . See Figure 5. Then V_{F_1, F_2} is a handlebody. Hence $V_{F_1, F_2} \cup_S W$ is also a Heegaard splitting.

Claim 3.4. The Heegaard distance of $V_{F_1, F_2} \cup_S W$, said $d_{\mathcal{C}(S)}(V_{F_1, F_2}, W)$, is n .

Proof. Suppose, otherwise, that $d_{\mathcal{C}(S)}(V_{F_1, F_2}, W) = k < n$. Since W contains only one essential disk D up to isotopy such that $\partial D = \beta$, there is an essential disk B_2 in V_{F_1, F_2} such that $d_{\mathcal{C}(S)}(\partial B_2, \beta) = k$, i.e., there is a geodesic $\mathcal{G} = \{a_0 = \beta, \dots, a_k = \partial B_2\}$, where $k \leq n - 1$. By the proof of Claim 1, $a_j \cap S_2 \neq \emptyset$ for $0 \leq j \leq k - 1$.

Note that $\partial B = \alpha$. Depending on the way of intersection between B_2 and B , either

(1) $B_2 \cap B = \emptyset$. Since $d_{\mathcal{C}(S)}(\alpha, \beta) = n > k$, B_2 is not isotopic to B . By the proof of Claim 1, ∂B_2 does not lie in S_1 . Hence $\partial B_2 \subset S_2$. It implies that $\psi_{F_2}(\partial B_2)$ bounds an essential disk in H_Y . By lemma 2.3, $d_{\mathcal{C}(S_2)}(\partial B_2, \beta) \leq \mathcal{M}$. Hence $d_{\mathcal{C}(F_2)}(\psi_{F_2}(\partial B_2), \psi_{F_2}(\beta)) \leq \mathcal{M}$, and $d_{\mathcal{C}(F_2)}(\mathcal{D}(H_Y), \psi_{F_2}(\beta)) \leq \mathcal{M}$. It contradicts the choice of Y . Or,

(2) $B_2 \cap B \neq \emptyset$. Let a^* be an outermost arc of $B_2 \cap B$ on B_2 . This means that a^* , together with a sub-arc $\gamma^* \subset \partial B_2$, bounds a disk B_{γ^*} such that $B_{\gamma^*} \cap B = a^*$. By the proof of Claim 4.1 (2), $\gamma^* \subset S_2$. Thus $\psi_{F_2}(\partial B_2)$ bounds an essential disk in H_Y . By the same argument in Claim 1, it is impossible. END. (Claim 2)

Until now, we get a distance n genus g Heegaard splitting $V_{F_1, F_2} \cup_S W$. In this case, V_{F_1, F_2} is a handlebody, and W contains only one essential disk D such that $\partial D = \beta$. Furthermore, we can cut S along β into two components S_3 and S_4 , and cut W along D into two manifolds $F_3 \times I$ and $F_4 \times I$ such that $F_i = F_i \times \{0\}$, and $S_i \cup D = F_i \times \{1\}$ for $i = 3, 4$. Now the shadow disk in Figure 3 is D . Let $f_{F_i} : S_i \cup D \rightarrow F_i$ be the natural homeomorphism such that $f_{F_i}(x \times \{1\}) = x \times \{0\}$ for $i = 3, 4$. Then, for any two essential simple closed curves $\zeta, \theta \subset S_i \cup D$, $d_{\mathcal{C}(F_i)}(f(\zeta), f(\theta)) = d_{\mathcal{C}(S_i \cup D)}(\zeta, \theta)$ for $i = 3, 4$, see Figure 3. Hence f_{F_i} induces an isomorphism from $\mathcal{C}(S_i \cup B)$ to $\mathcal{C}(F_i)$, for any $i = 1, 2$. Denote the isomorphism by f_{F_i} too.

Let $\iota : S_i \rightarrow S_i \cup D$ be the inclusion map for $i = 3, 4$. Note that ∂S_i contains only one component. If c is an essential simple closed curve in S_i , $\iota(c)$ is also essential in $S_i \cup D$. Now, for any two essential simple closed curves $\zeta, \theta \subset S_i$, $d_{\mathcal{C}(S_i \cup D)}(\iota(\zeta), \iota(\theta)) \leq d_{S_i}(\zeta, \theta)$ for $i = 3, 4$. Hence ι induces a distance non-increasing map from $\mathcal{C}(S_i)$ to $\mathcal{C}(S_i \cup B)$, for any $i = 1, 2$. Denote the inclusion map by ι too. Then we can define a projection map :

$$\psi_{F_i} = f_{F_i} \circ \iota \circ \pi_{S_i} : \mathcal{C}^0(S) \rightarrow \mathcal{C}^0(F_i).$$

Since $n \neq 2$, there are two cases:

Case 1. $n \geq 3$.

Since $V_{F_1, F_2} \cup_S W$ is a distance n Heegaard splitting of genus g , and W contains only an essential disk D up to isotopy, S_3 and S_4 are incompressible in V_{F_1, F_2} .

Hence $\beta = \partial S_3 = \partial S_4$ is disk-busting in V_{F_1, F_2} . Since $g \geq 3$, and $g(S_3) = 1$, V_{F_1, F_2} is not an I-bundle over some compact surface with S_i a horizontal boundary of the I-bundle, and the vertical boundary of this I-bundle a single annulus for $i = 3, 4$. By Lemma 2.4, $\text{diam}_{S_i}(\mathcal{D}(V_{F_1, F_2})) \leq 12$ for $i = 3, 4$. Hence $\text{diam}_{F_i}(\psi_{F_i}(\mathcal{D}(V_{F_1, F_2}))) \leq 12$.

Since F_3 is a torus, by Lemma 2.1, there is an essential simple closed curve δ in F_3 such that $d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1, F_2})), \delta) \geq \mathcal{M} + 1$. Let W_{F_3} be the manifold obtained attaching a solid J_δ to W along F_3 so that δ bounds a disk in J_δ . Then W_{F_3} is a compression body.

Since $g \geq 3$, $g(F_4) \geq 2$. By Lemma 2.5, there is a full simplex Z of $\mathcal{C}(F_4)$ such that $d_{\mathcal{C}(F_4)}(\mathcal{D}(H_Z), \psi_{F_4}(\mathcal{D}(V_{F_1, F_2}))) \geq \mathcal{M} + 1$, where H_Z is the handlebody obtained by attaching 2-handles to F_4 along Z then 3-handles to cap off the possible 2-spheres. In this case, let W_{F_3, F_4} be the handlebody $W_{F_3} \cup H_Z$. Now $V_{F_1, F_2} \cup_S W_{F_3, F_4}$ is a Heegaard splitting of a closed 3-manifold.

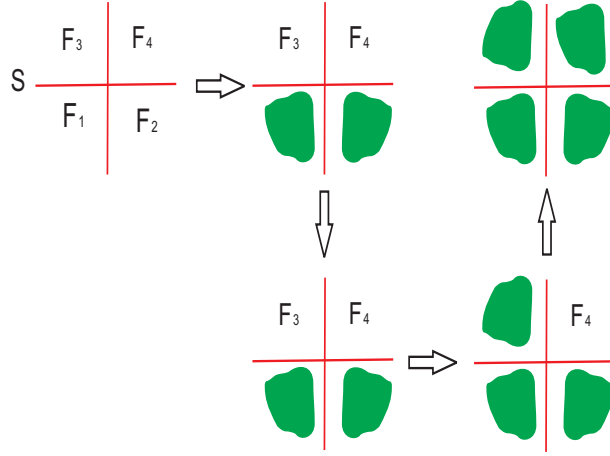


Figure 6

Claim 3.5. The distance of $V_{F_1, F_2} \cup_S W_{F_3, F_4}$, said $d_{\mathcal{C}(S)}(V_{F_1, F_2}, W_{F_3, F_4})$, is n .

Proof. Let D be the essential disk in W_{F_3, F_4} bounded by β . Suppose, otherwise, that $d = k < n$. Then there is a geodesic $\mathcal{G} = \{a_0 = \partial B_1, \dots, a_k = \partial D_1\}$, where $k \leq n - 1$, B_1 is a disk in V_{F_1, F_2} , and D_1 is a disk in W_{F_3, F_4} . $\alpha_i \cap \beta \neq \emptyset$, for any $0 \leq i \leq k - 1$ for if not, the distance of $V_{F_1, F_2} \cup_S W$ would be at most $k < n$. Similarly, D_1 is not isotopic to D .

Then either

(1) $D_1 \cap D = \emptyset$. Then ∂D_1 lies in one of S_3 and S_4 , say S_3 . Hence $\psi_{F_3}(\partial D_1) = \delta$. By Lemma 2.3, $\text{diam}_{S_3}(\mathcal{D}(\mathcal{G})) \leq \mathcal{M}$. Since $\pi_{S_3}(\partial B_1) \in \pi_{S_3}(\mathcal{D}(V_{F_1, F_2}))$, $d_{\mathcal{C}(S_3)}(\pi_{S_3}(\mathcal{D}(V_{F_1, F_2})), \partial D_1) \leq \mathcal{M}$. Hence $d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1, F_2})), \psi_{F_3}(\partial D_1) = \delta) \leq \mathcal{M}$, a contradiction. Or,

(2) $D_1 \cap D \neq \emptyset$. Let c be an outermost arc of $B_2 \cap B$ on B_2 . This means that c , together with a sub-arc $\delta^* \subset \partial B_2$, bounds a disk D_c such that $D_c \cap D = \gamma^*$. We may assume that $\partial D_c \subset S_4$. By Lemma 2.3, $\text{diam}_{S_4}(\mathcal{D}(\mathcal{G})) \leq \mathcal{M}$. Hence

$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\mathcal{D}(V_{F_1, F_2})), \psi_{F_4}(\partial B_2)) \leq \mathcal{M}$. Note that $\psi_{F_4}(\partial B_2) \in \mathcal{D}(H_Z)$. By the same argument in (1), it is impossible. END.

Now we suppose that $n = 1$.

Let M_1 and M_2 be two 3-manifolds with homeomorphic connected boundary. Let M^f be the manifold obtained by gluing M_1 and M_2 along a homeomorphism from ∂M_1 to ∂M_2 . Let $M_i = V_i \cup_{S_i} W_i$ be a minimal Heegaard splitting for $i = 1, 2$. In this case, M^f has a natural Heegaard called the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$. The following facts are well known:

- (1) If the gluing map f is enough complicated, then the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is unstabilized, see [2], [16], [21], [35].
- (2) If both $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ have high distance, then the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is unstabilized, See [15], [37].

Now let $M_i = V_i \cup_{S_i} W_i$ be a Heegaard splitting of genus two such that ∂M_i is a torus, and $d(S_i) > 8$ for $i = 1, 2$, then, by the main result in [15], the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$, say $V \cup_S W$, is unstabilized. Furthermore, $g(S) = 3$.

Suppose that $g \geq 4$. By the above argument, there are a Heegaard splitting $M_1 = V_1 \cup_{S_1} W_1$ of genus $g - 1$ such that $g(\partial M_1) = 2$, and $d(S_1) \geq 2g$, and a Heegaard splitting $V_2 \cup_{S_2} W_2$ of genus 3 such that $g(\partial M_2) = 2$, and $d(S_2) \geq 2g$. Hence both M_1 and M_2 are hyperbolic. By the main result in [15], the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$, say $M = V \cup_S W$, is unstabilized. Furthermore, $g(S) = g$. By Thurston's Theorem, M is hyperbolic.

END(Proposition 3.1)

Remark. The strongly irreducible Heegaard splitting $V \cup_S W$ where both V and W contain only one essential separating disk up to isotopy independently is always a minimal Heegaard splitting of $M = V \cup_S W$. T.Li [21] defined a sub-complex $\mathcal{U}(F_1)$, for $F_1 \subset \partial_- V$ and proved that for any handlebody H attached to M along F_1 , if $d_{\mathcal{C}(F_1)}(\mathcal{U}(F_1), \mathcal{D}(H))$ is larger than a constant \mathcal{K} which depends on M and H , then the new generated Heegaard splitting $V_{F_1} \cup_S W$ is still the minimal Heegaard splitting of $M^{F_1} = V_{F_1} \cup_S W$. Similar to the other boundaries of M . Now in our construction of distance $n \geq 2$ strongly irreducible Heegaard splitting (for $n=2$, see section 5), we can choose a full simplex X in F_1 such that $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\mathcal{D}(W)), \mathcal{D}(H_X))$ is large enough and $d_{\mathcal{C}(F_1)}(\mathcal{U}(F_1), \mathcal{D}(H_X))$ is larger than \mathcal{K} . Then the new Heegaard splitting $V_{F_1} \cup_S W$ is still the minimal Heegaard splitting of $M^{F_1} = V_{F_1} \cup_S W$ and has the same distance as the older one.

4. PROOF OF THEOREM 2

We will prove Theorem 2 in this section.

Theorem 2. For any integers $g \geq 2$ and $n \geq 4$, there are infinitely many non-homeomorphic closed 3-manifolds which admit distance n Heegaard splittings of genus g .

Proof. Let S_g be a closed surface of genus g . By Lemma 2.6, for each $m \geq 2$, there is a geodesic $\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \dots, a_{n-1}^m, a_n^m = \beta^m\}$ in $\mathcal{C}(S_g)$ such that

- (1) a_i is non-separating in S_g for $1 \leq i \leq n-1$, α and β^m are two essential separating simple closed curves on S_g for $m \geq 2$,
- (2) $m\mathcal{M} + 2 \leq d_{\mathcal{C}(S^{a_i})}(a_{i-1}, a_{i+1}) = m\mathcal{M} + 6$, where S^{a_i} is the surface $S - N(a_i)$ for $1 \leq i \leq n-1$, and
- (3) one component of $S_g - \beta^m$ has genus one.

Without loss of generality, we assume that $\mathcal{M} \geq 6$. Let M_m be the manifold obtained by attaching two 2-handles to $S_g \times [-1, 1]$ along $\alpha \times \{-1\}$ and $\beta^m \times \{1\}$. We denote also by S_g the surface $S_g \times \{0\}$. Now M_m has a Heegaard splitting as $V_m \cup_{S_g} W_m$, where V_m is the compression body obtained by attaching a 2-handle to $S \times [-1, 0]$ along $\alpha \times \{-1\}$, and W_m is the manifold obtained by attaching a 2-handle to $S \times [0, 1]$ along $\beta^m \times \{1\}$. Then $\partial_- V_m$ contains two components F_1 and F_2 , and $\partial_- W_m$ contains two components F_3^m and F_4^m . See Figure 7. Furthermore, one component of F_3^m and F_4^m has genus one.

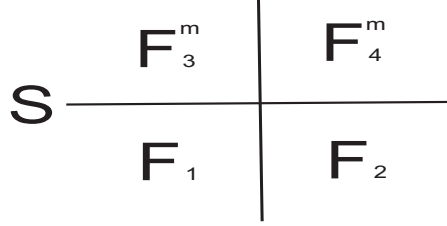


Figure 7

By the proof of Theorem 1(1), there is a closed 3-manifold M_m^* which admits a distance n Heegaard splitting $V_m^* \cup_{S_g} W_m^*$, where V_m^* is obtained by attaching handlebodies H_{X_1} and H_{X_2} to V_m along F_1 and F_2 , and W_m^* is obtained by attaching handlebodies H_{Y_1} and H_{Y_2} to W_m along F_3^m and F_4^m such that

- (1) $d_{\mathcal{C}(F_i)}(\psi_{F_i}(\beta^m), \mathcal{D}(H_{X_i})) \geq \mathcal{M} + 15$ for $i = 1, 2$, and
- (2) $d_{\mathcal{C}(F_i)}(\psi_{F_i}(\alpha), \mathcal{D}(H_{Y_i})) \geq \mathcal{M} + 15$ for $i = 3, 4$.

Replace M_m^* , V_m^* and W_m^* by M_m , V_m and W_m . Now $\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \dots, a_{n-1}^m, a_n^m = \beta^m\}$ is also a geodesic of $\mathcal{C}(S_g)$ realizing the distance of $M_m = V_m \cup_{S_g} W_m$.

Claim 4.2. Let $\mathcal{G} = \{b_0, \dots, b_n\}$ be a geodesic of $\mathcal{C}(S_g)$ realizing the distance of $V_m \cup_{S_g} W_m$. Then $b_i = a_i^m$ for any $1 \leq i \leq n-1$.

Proof. Let S_1 and S_2 be the two components of $S_g - \alpha$. We assume that b_0 bounds a disk B_0 in V_m , and b_n bounds a disk D_n in W_m . We first prove that α (resp. β^m) is disjoint from b_1 (resp. b_{n-1}).

Suppose, otherwise, that $\alpha \cap b_1 \neq \emptyset$. Hence b_0 is not isotopic to $a_0^m = \alpha$. Then either

- (1) $B_0 \cap B \neq \emptyset$. Let a be an outermost arc of $B_0 \cap B$ on B_0 . It means that a , together a sub-arc of $\gamma \subset \partial B_0$, bounds a disk B_γ such that $B_\gamma \cap B = a$. Without assumption, we may assume that $\gamma \subset S_1$. By the argument in section 3, $\psi_{F_1}(\partial B_0)$ bounds an essential disk in H_{X_1} . But with $b_1 \cap \partial S_1 \neq \emptyset$, it implies that $d_{\mathcal{C}(S_1)}(b_0, b_n) \leq \mathcal{M}$. Hence $d_{\mathcal{C}(F_1)}(\psi_{F_1}(b_n), \mathcal{D}(H_{X_1})) \leq \mathcal{M}$. Or,
- (2) $B_0 \cap B = \emptyset$. By $b_1 \cap \alpha \neq \emptyset$, B_0 is not isotopic to B . Then ∂B_0 is essential in S_1 or S_2 . We assume that $\partial B_0 \subset S_1$. The other case is similar. Hence by (1), $d_{\mathcal{C}(F_1)}(\psi_{F_1}(b_n), \mathcal{D}(H_{X_1})) \leq \mathcal{M}$.

However, by Heegaard distance is at least 4 and $\alpha = \partial S_1 = \partial S_2$ bounds an essential disk in V^m , it means that α is disk-busting for W^m and W^m can not be the I-bundle of compact surface with S_1 or S_2 as one of its horizontal boundary. Then by Lemma 2.4, $\text{diam}_{\mathcal{C}(S_1)}(\mathcal{D}(W^m)) \leq 12$ and $\text{diam}_{\mathcal{C}(S_2)}(\mathcal{D}(W^m)) \leq 12$. Hence $\text{diam}_{\mathcal{C}(F_1)}(\mathcal{D}(W^m)) \leq 12$ and $\text{diam}_{\mathcal{C}(F_2)}(\mathcal{D}(W^m)) \leq 12$. Together with (1) and (2), by triangle inequality, $d_{\mathcal{C}(F_1)}(\psi_{F_1}(\beta^m), \mathcal{D}(H_{X_1})) \leq \mathcal{M} + 12$. It contradicts the choice of X_1 in F_1 . The other case is similar.

Let $\mathcal{G}^* = \{\alpha = a_0^m, b_1, \dots, b_{n-1}, a_n^m\}$ be a new geodesic realizing the distance of $V_m \cup_{S_g} W_m$. Now we prove that b_1 is isotopic to a_1^m .

Suppose, otherwise, that b_1 is not isotopic to a_1^m . Note that b_i is not isotopic to a_1^m . Otherwise, the distance of $V_m^c \cup_{S_g} W_m^c$ would be at most $n - 1$. Let $S^{a_1^m}$ be the surface $S_g - N(a_1^m)$, where $N(a_1^m)$ is a open regular neighborhood of a_1^m on S_g . By Lemma 2.3, $d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_n^m)) \leq \mathcal{M}$. Now let's consider the shorter geodesic $\mathcal{G}^{**} = \{a_2^m, \dots, a_{n-1}^m, a_n^m = \beta^m\}$ which is a sub-geodesic of $\mathcal{G}^m = \{\alpha = a_0^m, a_1^m, \dots, a_{n-1}^m, a_n^m = \beta^m\}$. Due to the definition of geodesic in curve complex, a_i^m is not isotopic to a_1^m for any $i \geq 2$. By Lemma 2.3 again, $d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_2^m), \pi_{S^{a_1^m}}(a_n^m)) \leq \mathcal{M}$. Hence $d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_2^m)) \leq 2\mathcal{M}$. This contradicts our assumption on $d_{\mathcal{C}(S^{a_1^m})}(\pi_{S^{a_1^m}}(a_0^m), \pi_{S^{a_1^m}}(a_n^m))$. Hence b_1 is isotopic to a_1^m .

By induction on i , the claim holds. End (Claim 4.2)

Replace $M_m = V_m \cup_{S_g} W_m$ by $M_m = V_m \cup_{S_g^m} W_m$.

The following claim reveals the connection between geodesics in curve complex and closed 3-manifolds.

Claim 4.3. For any $2 \leq t \neq s \in N$, either

(1) $M_t = V_t \cup_{S_g^t} W_t$ and $M_s = V_s \cup_{S_g^s} W_s$ are two different 3-manifolds up to homeomorphism. Or,

(2) M_t is homeomorphic to M_s , but $V_t \cup_{S_g^t} W_t$ and $V_s \cup_{S_g^s} W_s$ are two different Heegaard splittings of M_t up to homeomorphic equivalence.

Proof. Suppose that M_t is homeomorphic to M_s for some $2 \leq t \neq s \in N$. If (2) fails, then $V_t \cup_{S_g^t} W_t$ and $V_s \cup_{S_g^s} W_s$ are homeomorphic. It means that there is a homeomorphism f from M_t to M_s such that $f((S_g^t; V_t, W_t)) = (S_g^s; V_t, W_t)$. We assume that $f(V_t) = V_s$ and $f(W_t) = W_s$. The other case is similar. It is well known that f induces an isomorphism from $\mathcal{C}(S_g^t)$ to $\mathcal{C}(S_g^s)$, still denoted by f . Then for the geodesic $\mathcal{G}^t = \{\alpha = a_0^t, a_1^t, \dots, a_{n-1}^t, a_n^t = \beta^t\}$ which realizes the distance of $V_t \cup_{S_g^t} W_t$, $f(\mathcal{G})$ is also a geodesic in $\mathcal{C}(S_g^s)$ realizing the distance of $V_s \cup_{S_g^s} W_s$. By Claim 4.2, $f(a_j^t)$ is isotopic to a_j^s for $1 \leq j \leq n - 1$.

As $n \geq 4$, we choose a_2^t . Since $f(a_2^t)$ is isotopic to a_2^s , we can perform an isotopy on S_g^s such that the composition of f with the isotopy gives an homeomorphism f^* from S_t to S_s and $f^*(a_2^t) = a_2^s$. Even more, $f^*(V_t) = V_s$ and $f^*(W_t) = W_s$. It's still true that f^* induces an automorphism from $\mathcal{C}(S_g^t)$ to $\mathcal{C}(S_g^s)$, denoted by f^* too. Thus $f^*(\mathcal{G}^t)$ is also a geodesic realizing the distance of $V_s \cup_{S_g^s} W_s$. By Claim 4.2 again, for any $1 \leq j \leq n - 1$, $f^*(a_j^t)$ is still isotopic to a_j^s . Hence $f^*(a_1^t)$ (resp. $f^*(a_3^t)$) is isotopic to a_1^s (resp. a_3^s).

Let $S^{a_2^t}$ be the surface $S_g^t - N(a_2^t)$, where $N(a_2^t)$ is an open regular neighborhood of a_2^t on S_g^t , and $S^{a_2^s}$ be the surface of $S_g^s - N(a_2^s)$. Then $f^*(S^{a_2^t}) = S^{a_2^s}$ and $f^*|_{S^{a_2^t}}$ is a homeomorphism. Hence f^* also induces an isomorphism from $\mathcal{C}(S^{a_2^t})$ to $\mathcal{C}(S^{a_2^s})$, still denoted by f^* . Now we can also assume $a_1^t \cap a_2^t = \emptyset$ and $a_3^t \cap a_2^t = \emptyset$. Thus $f^*(a_1^t) \cap (f^*(a_2^t) = a_2^s) = \emptyset$ and $f^*(a_3^t) \cap (f^*(a_2^t) = a_2^s) = \emptyset$. Then $d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) = d_{\mathcal{C}(S^{a_2^s})}(f^*(a_1^t), f^*(a_3^t))$. On the other side, $f^*(a_1^t)$ (resp. $f^*(a_3^t)$) must be isotopic to a_1^s (resp. a_3^s) in $S^{a_2^s}$ for \triangleright if not, then after removing possible Bigon capped by them, they bounds no annulus in $S^{a_2^s}$, thus they bounds no annulus and Bigon in S_g^s . By Bigon Criterion (proposition 1.7[7]), they realizes the geometry intersection number. Since they are isotopic in S_g^s , they must be disjoint in S_g^s . Hence they must bounds an annulus in $S_g^s \triangleleft$. So $d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) = d_{\mathcal{C}(S^{a_2^s})}(f^*(a_1^t), f^*(a_3^t)) = d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s)$. However, by the assumption $[t\mathcal{M} + 2 \leq d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) \leq t\mathcal{M} + 6, s\mathcal{M} + 2 \leq d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s) \leq s\mathcal{M} + 6$ and $\mathcal{M} \geq 6]$, $d_{\mathcal{C}(S^{a_2^t})}(a_1^t, a_3^t) \neq d_{\mathcal{C}(S^{a_2^s})}(a_1^s, a_3^s)$, a contradiction. End (Claim 4.3)

The Waldhausen conjecture proved by Johanson ([12],[13]) and Li [19, 20] implies that, for any positive integer g , an atoroidal closed 3-manifold M admits only finitely many Heegaard splittings of genus g up to homeomorphism. Since M_t admits a Heegaard splitting with distance at least 4, it is atoroidal for any $t \geq 2$, see [10] and [33]. Now Theorem 2 is immediately from Claim 2 and the Waldhausen conjecture. END

5. PROOF OF THEOREM 1(2)

We rewrite the second part of Theorem 1 as the following proposition:

Proposition 5.1. For any integer $g \geq 2$, there is a hyperbolic closed 3-manifold which admits a distance 2 Heegaard splitting of genus g .

Proof. By the remark on Theorem 1, there is a hyperbolic closed 3-manifold which admits a distance 2 Heegaard splitting of genus 2.

Suppose now that $g \geq 3$.

Assumption 1. Let S be a closed surface of genus g . By Lemma 2.6, there are two separating slopes α and γ such that

- (1) $d_{\mathcal{C}(S)}(\alpha, \gamma) = 2$,
- (2) one component of $S - \alpha$, say S_1 , has genus one while another component of $S - \alpha$, say S_2 , has genus $g - 1$,
- (3) one component of $S - \gamma$, say S_3 , has genus one, while another component of $S - \gamma$, say S_4 , has genus $g - 1$
- (4) there is a non-separating slope β on S such that α and γ are disjoint from β , and $d_{\mathcal{C}(S^\beta)}(\alpha, \gamma) > 4$, where S^β is the surface $S - \eta(\beta)$, and
- (5) $\beta \subset S_2 \cap S_4$.

Let V be the compression body obtained by attaching a separating 2-handle to $S \times [0, 1]$ along $\alpha \times \{1\}$, and W be the compression body obtained by attaching a separating 2-handle to $S \times [-1, 0]$ along $\gamma \times \{-1\}$. Denote $S \times \{0\}$ by S too. Then $V \cup_S W$ is a Heegaard splitting. Since V contains only one essential disk B with

$\partial B = \alpha$ up to isotopy, and W contains only one essential disk D with $\partial D = \gamma$ up to isotopy, $d_{\mathcal{C}(S)}(V, W) = 2$.

Let F_1 and F_2 be the components of $\partial_- V$, such that F_i is homeomorphic to $S_i \cup B$ for $i = 1, 2$. Similarly, let F_3 and F_4 be the components of $\partial_- W$ such that F_i is homeomorphic to $S_i \cup D$ for $i = 3, 4$. Then both S_1 and S_3 are once-punctured tori, and F_1 and F_3 are two tori, see Figure 2. Furthermore, both F_3 and F_4 have genus at least 2. Now B cuts V into two manifolds $F_1 \times I$ and $F_2 \times I$, and D cuts W into two manifolds $F_3 \times I$ and $F_4 \times I$.

Since $d_{\mathcal{C}(S)}(V, W) = 2$, $\gamma \cap S_i \neq \emptyset$ for $i = 1, 2$, and $\alpha \cap S_i \neq \emptyset$ for $i = 3, 4$. Hence $\psi_{F_i}(\gamma) \neq \emptyset$ for $i = 1, 2$, and $\psi_{F_i}(\alpha) \neq \emptyset$ for $i = 3, 4$; where ψ is defined in Section 3.

Assumption 2. (1) Let δ be an essential simple closed curve on the torus F_1 such that $d_{\mathcal{C}(F_2)}(\psi_{F_2}(\gamma), \delta) \geq 5$

(2) Let X be a full complex of $\mathcal{C}(F_2)$ such that $d_{\mathcal{C}(F_2)}(\psi_{F_2}(\gamma), \mathcal{D}(H_X)) \geq 24$, where H_X is the handlebody obtained by attaching 2-handles to F_2 along the vertices of X then 3-handles to capping off the spherical boundary components.

Let $V_{F_2} = V \cup H_X$, and V_{F_1, F_2} be the handlebody obtained by doing a surgery on V_{F_2} along the slope δ on F_1 . By Assumption 1, $g(S_3) = 1$, $g(S_4) \geq 2$, V_{F_1, F_2} is not a I -bundle over a compact surface with S_i as a horizontal boundary for $i = 3, 4$. By Lemma 2.4, $\text{diam}_{\mathcal{C}(S_i)}(\pi_{S_i}(\mathcal{D}(V_{F_1, F_2}))) \leq 12$ for $i = 3, 4$.

Assumption 3. (1) Let r be an essential simple closed curve on the torus F_3 such that $d_{\mathcal{C}(F_3)}(\psi_{F_3}(\mathcal{D}(V_{F_1, F_2})), r) \geq 24$.

(2) Let Y be a full complex of $\mathcal{C}(F_4)$ such that $d_{\mathcal{C}(F_4)}(\psi_{F_4}(\mathcal{D}(V_{F_1, F_2})), \mathcal{D}(H_Y)) \geq 24$, where H_Y is the handlebody obtained by attaching 2-handles to F_4 along the vertices of Y then 3-handles to capping off the spherical boundary components.

Let $W_{F_4} = W \cup H_Y$, and W_{F_3, F_4} be the handlebody obtained by doing a surgery on W_{F_4} along the slope r on F_3 . Now both $M^* = V_{F_2} \cup_S W_{F_4}$ and $V_{F_1, F_2} \cup_S W_{F_3, F_4}$ are Heegaard splittings. Furthermore, we can prove that these two Heegaard splittings have distance 2 by using Lemma 2.2 to take place of Lemma 2.3 in the proof of Proposition 3.1.

Now we consider $M^* = V_{F_2} \cup_S W_{F_4}$. Note that M^* has only two toral components. Since the distance of $V_{F_2} \cup_S W_{F_4}$ is 2, M^* is irreducible and ∂ -irreducible.

Claim 1. M^* is atoroidal.

Proof. Suppose, otherwise, that M^* contains an essential torus T . Since the distance of $V_{F_2} \cup_S W_{F_4}$ is 2, $V_{F_2} \cup_S W_{F_4}$ is strongly irreducible. By Schultens's lemma, each component of $T \cap S$ is essential on both T and S . Hence each component of $T \cap V_{F_2}$ and $T \cap W_{F_4}$ is an essential annulus in V_{F_2} or W_{F_4} .

Let A_0 be one component of $T \cap V_{F_2}$. We first prove that there is one component of ∂A_0 , say a_0 , is not isotopic to β .

Now V_{F_2} contains a ∂ -compressing disk B^* of A_0 . By doing a surgery on A_0 along B^* , we can get a disk B_0 in V_{F_2} . Since A_0 is essential, B_0 is essential. Suppose that the two components of ∂A_0 are isotopic to β . Since β is non-separating on S , ∂B_0 bounds a once-punctured torus containing β , see Figure 8.

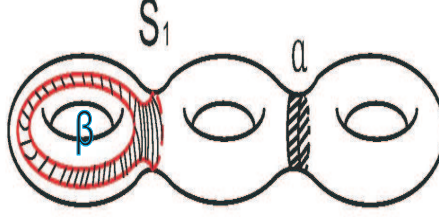


Figure 8

By Assumption 1, $\beta \subset S_2$. Since S_2 has genus $g - 1 \geq 2$, ∂B_0 is not isotopic to $\alpha = \partial S_2$. By a standard outermost argument, $\psi_{F_2}(\partial B_0)$ bounds an essential disk in H_X . Therefore $d_{C(F_2)}(\mathcal{D}(H_X), \psi_{F_2}(\beta)) \leq 1$. Since $\gamma \cap \beta = \emptyset$, $d_{C(F_2)}(\psi_{F_2}(\beta), \psi_{F_2}(\gamma)) \leq 1$. Hence $d_{C(F_2)}(\mathcal{D}(H_X), \psi_{F_2}(\gamma)) \leq 2$. It contradicts Assumption 2.

Let A_1 be a component of $T \cap W_{F_4}$ which is incident to A_0 . This means that a_0 is one component of ∂A_1 .

Case 1. $a_0 \cap \alpha = \emptyset$, and $a_0 \cap \gamma = \emptyset$.

Recall the definition of the surface S^β . Since a_0 is not isotopic to β , $a_0 \cap S^\beta \neq \emptyset$. Since $\alpha, \gamma \subset S^\beta$, $d_{C(S^\beta)}(\pi_{S^\beta}(a_0), \alpha) \leq 1$, and $d_{C(S^\beta)}(\gamma, \pi_{S^\beta}(a_0)) \leq 1$. Hence $d_{C(S^\beta)}(\alpha, \gamma) \leq 2$. This contradicts Assumption 1.

Case 2. $a_0 \cap (\alpha \cup \gamma) \neq \emptyset$.

We assume that $a_0 \cap \alpha \neq \emptyset$. By the above argument, B_0 is an essential disk in V_{F_2} such that ∂B_0 is disjoint from a_0 . Furthermore, ∂B_0 is not isotopic to α . Since B cuts V_{F_2} into $F_1 \times I$ and a handlebody H such that $S_2 \cup B = \partial H$, $\partial B_0 \cap S_2 \neq \emptyset$. Furthermore, all outermost disks of $B_0 \cap B$ on B_0 lie in H . Hence $\pi_{S_2}(\partial B_0)$ bounds an essential disk in H . This means $\psi_{F_2}(\partial B_0)$ bounds an essential disk in H_X .

If $a_0 \cap \gamma = \emptyset$, then

$$\begin{aligned} d_{C(F_2)}(\psi_{F_2}(\partial B_0), \psi_{F_2}(\gamma)) &\leq \\ d_{C(F_2)}(\psi_{F_2}(\partial B_0), \psi_{F_2}(a_0)) + d_{C(F_2)}(\psi_{F_2}(a_0), \psi_{F_2}(\gamma)) &\leq 4. \end{aligned}$$

It contradicts Assumption 2. Hence $a_0 \cap \gamma \neq \emptyset$, and $\psi_{F_4}(a_0) \neq \emptyset$.

Since A_1 is an essential annulus in W_{F_4} , there is an essential disk D_0 obtained by doing boundary compression on A_1 in W_{F_4} . Even more $\partial D_0 \cap a_0 = \emptyset$. Since D cuts W_{F_4} into $F_3 \times I$ and a handlebody H^* containing H_Y , all outermost disks of $D_0 \cap D$ in D_0 lie in H^* . Hence $\psi_{F_4}(\partial D_0)$ bounds an essential disk in H_Y . Hence $\pi_{S_4}(\partial D_0) \neq \emptyset$. Since $\partial D_0 \cap a_0 = \emptyset$, by Lemma 2.2, $d_{C(S_4)}(\pi_{S_4}(\partial D_0), \pi_{S_4}(a_0)) \leq 2$. According to the definition of ψ_{F_4} , $d_{C(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(a_0)) \leq 2$.

Recall that the essential disk B_0 is obtained by doing a surgery on A_0 along a ∂ -compressing disk in V_{F_2} . Since the distance of $V_{F_2} \cup_S W_{F_4}$ is two, $\partial B_0 \cap \gamma \neq \emptyset$. Since $g(S_3) = 1$ and $g(S_4) \geq 2$, V_{F_2} is not a I-bundle of compact surface with S_4

as one horizontal boundary, by Lemma 2.4, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial B_0), \pi_{S_4}(\alpha)) \leq 12$. Hence $d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(\alpha)) \leq 12$. Since $\partial B_0 \cap a_0 = \emptyset$, $d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(a_0)) \leq 2$. It means that

$$\begin{aligned} d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(\alpha)) &\leq d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(a_0)) + \\ d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(a_0)) &+ d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial B_0), \psi_{F_4}(\alpha)) \leq 16. \end{aligned}$$

It contradicts Assumption 3. END(Claim 1)

Claim 2. M^* is anannular.

Proof. Suppose, otherwise, that M^* contains an essential torus A . Since the distance of $M^* = V_{F_2} \cup_S W_{F_4}$ is 2, $M^* = V_{F_2} \cup_S W_{F_4}$ is strongly irreducible. By Schultens's lemma, each component of $A \cap S$ is essential on both A and S . Hence each component of $A \cap V_{F_2}$ and $A \cap W_{F_4}$ is either a spanning annulus or an essential annulus with two boundary components lying on S . There are four cases:

Case 1. $|A \cap S| \geq 4$.

In this case, let A_0 be one component of $A \cap V_{F_2}$ and $A \cap W_{F_4}$ such that each of the two components of $A \cap V_{F_2}$ and $A \cap W_{F_4}$ incident to A_0 has its two boundary components lying on S . By the proof of Claim 1, the claim holds.

Case 2. $|A \cap S| = 1$.

Now A intersects S in an essential simple closed curve a . Furthermore, a , together with an essential simple closed curve c_1 on F_1 , bounds a spanning annulus A_1 in V_{F_2} , and a , together with an essential simple closed curve c_2 on F_3 , bounds a spanning annulus A_2 in W_{F_4} . Hence $a = c_1$ in $H_1(V_{F_2})$. Since B cuts V_{F_2} into $F_1 \times I$ and a handlebody H containing β , $a \neq \beta$ in $H_1(V_{F_2})$. Hence a is not isotopic to β . There are three sub-cases:

Case 2.1. $a \cap \alpha = \emptyset$, and $a \cap \gamma = \emptyset$.

In this case, $a \subset S_1, S_3$, and $\beta \subset S_2, S_4$, Hence $a \cap \beta = \emptyset$. This means that $d_{\mathcal{C}(S^\beta)}(\alpha, \gamma) \leq 2$. It contradicts Assumption 1.

Case 2.2. $a \cap \alpha = \emptyset$, and $a \cap \gamma \neq \emptyset$.

Now $A_2 \cap D \neq \emptyset$. Let c be an outermost arc of $A_2 \cap D$ on D . This means that c , together with a sub-arc c^* of $\gamma = \partial D$, bounds a disk D^* in D such that $D^* \cap D = c^*$. Now we can obtain a disk D_0 by doing a surgery on A_2 along D^* , say D_0 . Hence $\partial D_0 \cap a = \emptyset$. Furthermore, D_0 is an essential disk in W_{F_4} . Otherwise, we can reduce $|A_2 \cap \gamma|$. Since D cuts W_{F_4} into $F_3 \times I$ and a handlebody H^* containing S_4 . Hence $\partial D_0 \cap S_4 \neq \emptyset$. Furthermore, one outermost disk of $D_0 \cap D$ on D_0 lies in H^* . Otherwise, we can reduce $|A_2 \cap \gamma|$. Hence $\psi_{F_4}(\partial D_0)$ bounds an essential disk in H_Y . By the assumption, $a \cap \gamma \neq \emptyset$. Hence $a \cap S_4 \neq \emptyset$. Since the distance of $V_{F_2} \cup_S W_{F_4}$ is 2, $\alpha \cap S_4 \neq \emptyset$. Now by Lemma 2.2 and the above argument,

$$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(\alpha)) \leq d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(a)) + d_{\mathcal{C}(F_4)}(\psi_{F_4}(a), \psi_{F_4}(\alpha)) \leq 4.$$

It contradicts Assumption 3.

Case 2.3. $a \cap \alpha \neq \emptyset$, and $a \cap \gamma \neq \emptyset$.

Let D_0 be as in Case 2.2. Since $a \cap \alpha \neq \emptyset$, $A_1 \cap B \neq \emptyset$. Let b be an outermost arc of $A_1 \cap B$ on B . This means that b , together with a sub-arc b^* of α , bounds a disk B^* in B such that $B^* \cap B = b^*$. Now we can obtain a disk B_0 by doing a surgery on A_1 along B^* , say B_0 . Hence $\partial B_0 \cap a = \emptyset$. Similarly, B_0 is an essential

disk in V_{F_2} . Since the distance of $V_{F_2} \cup_S W_{F_4}$ is 2, $B_0 \cap S_4 \neq \emptyset$. By Lemma 2.4, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial B_0), \pi_{S_4}(\alpha)) \leq 12$. By Lemma 2.2, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial B_0), \pi_{S_4}(a)) \leq 2$. By the argument in Case 2.2, $d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial D_0), \pi_{S_4}(a)) \leq 2$. Now we have

$$d_{\mathcal{C}(F_4)}(\psi_{F_4}(\partial D_0), \psi_{F_4}(\alpha)) \leq d_{\mathcal{C}(S_4)}(\pi_{S_4}(\partial D_0), \pi_{S_4}(\alpha)) \leq 16.$$

Note that $\psi_{F_4}(\partial D_0)$ is an essential disk in H_Y . It contradicts Assumption 3.

Case 3. $|A \cap S| = 2$.

Now we may assume that $A \cap V_{F_2}$ contains two spanning annulus, and $A \cap W_{F_2}$ is an annulus with its two boundary components lying on S . By the arguments in Claim 1 and Case 2, Claim 2 holds.

Case 4. $|A \cap S| = 3$.

This case immediately from Claim 1 and Case 3. END(Claim 2)

Now M^* is a hyperbolic 3-manifold, $M^* = V_{F_2} \cup_S W_{F_4}$ is a distance 2 Heegaard splitting of genus g . Furthermore, M^* contains two toral boundary components F_1 and F_3 . By the main results in [1] and [16], there are at most ten slopes δ on F_1 such that the manifold $M^*(\delta)$ obtained by doing Dehn filling on M^* along δ is non-hyperbolic. By Assumption 2, there are infinitely many slopes δ so that $M^*(\delta)$ has a distance 2 Heegaard splitting of genus g . Hence there is at least one slope δ on F_1 such that $M^*(\delta)$ is hyperbolic and $M^*(\delta)$ admits a distance 2 Heegaard splitting of genus g . Similarly, by Assumption 3, there is a hyperbolic closed manifold which admits a distance 2 Heegaard splitting of genus g . END

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